Contemporary Mathematics
Volume **238**, 1999
B 0-8218-1196-7-03551-9

Local Estimates for Solutions to Singular and Degenerate Quasilinear Parabolic Equations

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1. Introduction and Results

We shall obtain $L_{q,loc}(\Omega_T)$ and $L_{\infty,loc}(\Omega_T)$ estimates for a class of equations modeled after

$$u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,t) + \operatorname{div}\mathbf{g}. \tag{1}$$

If $p > 2$ the equation is degenerate, while if $p < 2$ the problem is singular. In particular, we shall study solutions of equations of the form

$$u_t - \operatorname{div} a(x,t,u,\nabla u) = b(x,t,u,\nabla u) \tag{2}$$

on domains $\Omega_T = \Omega \times (0,T)$ where $\Omega \subset \mathbf{R}^N$ and the equation satisfies the following structure conditions for each $(x,t,u,\mathbf{v}) \in \Omega \times (0,T) \times \mathbf{R} \times \mathbf{R}^N$

(H1) $1 < p \le \delta < p\left(\frac{N+2}{N}\right) \equiv m$, $c_i \ge 0$ for $0 \le i \le 5$, $c_0 > 0$, and $\phi_j \ge 0$ for $0 \le j \le 2$,
(H2) $a(x,t,u,\mathbf{v})\cdot\mathbf{v} \ge c_o|\mathbf{v}|^p - c_3|u|^\delta - \phi_o(x,t)$,
(H3) $|a(x,t,u,\mathbf{v})| \le c_1|\mathbf{v}|^{p-1} + c_4|u|^{\delta(1-\frac{1}{p})} + \phi_1(x,t)$,
(H4) $|b(x,t,u,\mathbf{v})| \le c_2|\mathbf{v}|^{p(1-\frac{1}{\delta})} + c_5|u|^{\delta-1} + \phi_2(x,t)$,
(H5) $\phi_1 \in L_{\frac{p}{p-1},loc}(\Omega_T)$,
(H6) $\phi_o \in L_{\mu,loc}(\Omega_T)$ with $\mu > 1$, and $\phi_1, \phi_2 \in L_{s,loc}(\Omega_T)$ with $s > \frac{m}{m-1}$,

while on the solution u we assume

(H7) For every $0 \le t_1 < t_2 \le T$ and for every $\Omega' \Subset \Omega$

$$\operatorname*{ess\,sup}_{t_1<t<t_2} \int_{\Omega'} |u(x,t)|^2\,dx + \int_{t_1}^{t_2}\int_{\Omega'} |\nabla u|^p\,dx\,dt < \infty,$$

(H8) $u \in L_{r,loc}(\Omega_T)$ for some $r > \frac{N}{p}(2-p)$.

By a weak solution of (2) we mean a function u that satisfies H8 and for which

$$\iint_{\Omega_T} \{-u\psi_t + a(x,t,u,\nabla u)\cdot\nabla\psi\}\,dx\,dt = \iint_{\Omega_T} b(x,t,u,\nabla u)\psi\,dx\,dt \tag{3}$$

for all $\psi \in C_0^\infty(\Omega_T)$.

Our main result is the following.

1991 *Mathematics Subject Classification.* Primary: 35K65, 35B45.

THEOREM 1. *Let u be a weak solution of* (2), *and suppose that H1-H8 are satisfied.*

If $\min\{s,\mu\} > (N+p)/p$, *then* $u \in L_{\infty,loc}(\Omega_T)$;
if $\min\{s,\mu\} = (N+p)/p$, *then* $u \in L_{q,loc}(\Omega_T)$ *for all* $q < \infty$;
if $\min\{s,\mu\} < (N+p)/p$, *then* $u \in L_{q,loc}(\Omega_T)$ *for all* $q < q^*$, *where*

$$q^* = \min\left\{ \frac{m - (1+\frac{p}{N})}{1-(1-\frac{1}{s})(1+\frac{p}{N})}, \frac{m}{1-(1-\frac{1}{\mu})(1+\frac{p}{N})} \right\}. \tag{4}$$

Moreover, the resulting bounds are independent of $\|\phi_1\|_{L_{\frac{p}{p-1},loc}(\Omega_T)}$.

Regularity properties of solutions of these types of equations have been extensively studied; an excellent reference is the book of DiBenedetto [**5**]. More specifically, Hölder continuity of solutions was proven in the degenerate case by DiBenedetto and Friedman [**6, 7**], while in the singular case by Y.Z. Chen and DiBenedetto in [**3, 4**]. Local boundedness of solutions under appropriate structure conditions was proven by Porzio [**14**] and these results have been extended to equations with more general structure in [**1, 8, 9, 11, 12, 15, 17, 18**].

The results contained in this paper have the following new features. First, to the best of this author's knowledge, this is the only result which yields information about the degree of local integrability of solutions which are not necessarily bounded. Secondly, this result extends the class of equations for which the local boundedness of solutions is guaranteed. Indeed, for the case $p > \frac{2N}{N+2}$, in [**5**, Chp. 5, Thm 3.1] boundedness of solutions was proven only if

$$\phi_1^{\frac{p}{p-1}}, \phi_2^{\frac{\delta}{\delta-1}} \in L_{s,loc}(\Omega_T) \qquad \text{for } s > \frac{N+p}{p}. \tag{5}$$

In the case $p \leq \frac{2N}{N+2}$, local boundedness was proven in [**5**, Chp. 5, Thm. 5.1] only if the problem had homogeneous structure, meaning (H2), (H3) and (H4) are replaced by the requirements $a(x,t,u,\mathbf{v})\cdot\mathbf{v} \geq c_o|\mathbf{v}|^p$, $|a(x,t,u,\mathbf{v})| \leq c_1|\mathbf{v}|^{p-1}$ and $b(x,t,u,\mathbf{v}) = 0$; moreover further global information was required, to the effect that the solution could be approximated weakly in $L_{r,loc}(\Omega_T)$ by bounded solutions. Only under these additional conditions, now no longer necessary, was boundedness proven.

We remark that the results of this note are almost optimal in the sense that they almost agree with the results of the linear case ($p = 2$). In particular, in [**10**, Chp. 3, Secs. 8,9] it is shown that solutions of linear problems of the form

$$u_t - \{a^{ij}(x,t)u_{x_j} + a^i(x,t)u\}_{x_i} + b^i(x,t)u_{x_i} + a(x,t)u = \phi(x,t) + \phi^i_{x_i} \tag{6}$$

when $\phi \in L_{s,loc}(\Omega_T)$ and $\phi^i \in L_{\mu,loc}(\Omega_T)$ are in $L_\infty(\Omega_T)$ when $\min\{s,\mu\} > (N+p)/p$, while they are in $L_{q,loc}(\Omega_T)$ for all $q < \infty$ if $\min\{s,\mu\} = (N+p)/p$, and are in $L_{q^*,loc}(\Omega_T)$ otherwise, where q^* is the number in Theorem 1 with $p = 2$.

A few comments on our hypotheses are now in order. The assumption H5 is made only to ensure that terms of the form $a(x,t,u,\nabla u)\cdot\nabla u$ are integrable. This information is needed only qualitatively and the resulting bounds are independent of $\|\phi_1\|_{\frac{p}{p-1}}$. The restriction on s in H6 is exactly that which is needed to ensure that $q^* > m$; recall that H7 and the Sobolev embedding theorem will imply that $u \in L_{m,loc}(\Omega_T)$. Finally, it is noted in [**5**] that the requirement H8 is necessary to prove boundedness of the solutions.

2. Proof of the $L_{q,loc}(\Omega_T)$ Estimates for $q < \infty$

The first step in our proof is the following local energy estimate.

Proposition 2. *Suppose that u is a solution of (2) and that H1-H8 are satisfied. Then for any $Q_R(x_o,t_o) \equiv B_R(x_o) \times (t_o - R^p, t_o) \Subset \Omega_T$, for any $0<\sigma<1$, and for any $k>0$ we have*

$$
\begin{aligned}
\left[\iint_{Q_{\sigma R}} (u \mp k)_\pm^m \, dx\, dt\right]^{\frac{1}{1+p/N}} &\le \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_R} (u\mp k)_\pm^2 \, dx\, dt \\
&+ \frac{\gamma}{(1-\sigma)^p R^p} \iint_{Q_R} (u\mp k)_\pm^p \, dx\, dt + \gamma \iint_{Q_R} |u|^\delta \chi[(u\mp k)_\pm > 0] \, dx\, dt \\
&+ \gamma \left[\frac{\|\phi_1\|_{L_s(Q_R)}}{(1-\sigma)R} + \|\phi_2\|_{L_s(Q_R)}\right] \left[\iint_{Q_R} (u\mp k)_\pm^{\frac{s}{s-1}} dx\, dt\right]^{1-\frac{1}{s}} \\
&+ \gamma \|\phi_o\|_{L_\mu(Q_R)} (\text{meas}[(u\mp k)_\pm > 0])^{1-\frac{1}{\mu}}
\end{aligned}
\tag{7}
$$

where γ depends only on c_i, N, p, δ, s and μ, but is independent of k.

This is a standard result proven by using a smooth cutoff approximation of $(u \mp k)_\pm$ as a testing function; for details see [**13**] or [**5**, Chp. 5, Prop. 6.1].

Our plan is to start with the assumption that $u \in L_{\beta,loc}(\Omega_T)$ for some $\beta \ge m$. We shall then estimate (7) in terms of $\|u\|_{L_\beta(Q_R)}$ and powers of k. This will give us an estimate of the form $|u|_{L^{\text{weak}}_{\alpha(\beta)}(Q_{\sigma R})} \le C$ for some function $\alpha(\beta)$, which will give us our $L_{q,loc}(\Omega_T)$ estimates for $q < \infty$.

Indeed, recall that a measurable function u is an element of $L_q^{\text{weak}}(\mathcal{U})$ if and only if

$$
|u|^q_{L_q^{\text{weak}}} \equiv \sup_{k>0} k^q \,\text{meas}[|u| > k] < \infty. \tag{8}
$$

Moreover, $L_q(\mathcal{U}) \subset L_q^{\text{weak}}(\mathcal{U}) \subset L_{q'}(\mathcal{U})$ for all $q' < q$ provided $\mathcal{U}$ is bounded. More details about the spaces $L_q^{\text{weak}}(\mathcal{U})$ can be found in [**2**, Chp. 1] or [**16**, IX.4].

As a consequence, our knowledge that $u \in L^{\text{weak}}_{\alpha(\beta),loc}(\Omega_T)$ lets us conclude that $u \in L_{q,loc}(\Omega_T)$ for all $q < \alpha(\beta)$. Repeating this process then tells us that $u \in L_{q,loc}(\Omega_T)$ for all $q < (\alpha \circ \alpha \circ \cdots \circ \alpha)(\beta)$. Carefully calculating $\alpha(\beta)$ and iterating shall then give us our $L_{q,loc}(\Omega_T)$ estimates.

Indeed, estimate the left side of (7) with the $(u-k)_+$ choice as

$$
\iint_{Q_{\sigma R}} (u-k)_+^m \, dx\, dt \ge k^m \underset{Q_{\sigma R}}{\text{meas}}[u > 2k]. \tag{9}
$$

On the other hand, if $\theta < \beta$ then

$$
\begin{aligned}
\iint_{Q_R} (u-k)_+^\theta \, dx\, dt &\le \left[\iint_{Q_R} (u-k)_+^\beta \, dx\, dt\right]^{\frac{\theta}{\beta}} (\text{meas}[u>k])^{1-\frac{\theta}{\beta}} \\
&\le \|u\|^\theta_{L_\beta(Q_R)} \left[\frac{1}{k^\beta} |u|^\beta_{L_\beta^{\text{weak}}(Q_R)}\right]^{1-\frac{\theta}{\beta}} \le \|u\|^\beta_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{\beta-\theta}.
\end{aligned}
\tag{10}
$$

Using this in (7) then gives us an estimate of the form

$$\left(k^m \operatorname*{meas}_{Q_{\sigma R}}[u > 2k]\right)^{\frac{1}{1+p/N}} \leq \gamma \|u\|^{\beta}_{L_\beta(Q_R)} \left[\left(\frac{1}{k}\right)^{\beta-2} + \left(\frac{1}{k}\right)^{\beta-p} + \left(\frac{1}{k}\right)^{\beta-\delta}\right] + \gamma \|u\|^{\beta(1-\frac{1}{s})}_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{\beta(1-\frac{1}{s})-1} + \gamma \|u\|^{\beta(1-\frac{1}{\mu})}_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{\beta(1-\frac{1}{\mu})} \tag{11}$$

for a γ that also depends on σ, R, $\|\phi_o\|_{L_\mu(Q_R)}$ and $\|\phi_1, \phi_2\|_{L_s(Q_R)}$. If we repeat this process for $(u+k)_-$, we obtain the estimate

$$\operatorname*{meas}_{Q_{\sigma R}}[u > k] \leq \gamma \left[\left(\frac{1}{k}\right)^{(\beta-2)(1+\frac{p}{N})+m} + \left(\frac{1}{k}\right)^{(\beta-\delta)(1+\frac{p}{N})+m} + \left(\frac{1}{k}\right)^{[\beta(1-\frac{1}{s})-1](1+\frac{p}{N})+m} + \left(\frac{1}{k}\right)^{\beta(1-\frac{1}{\mu})(1+\frac{p}{N})+m}\right] \tag{12}$$

for all $k \geq 1$, where γ now also depends on $\|u\|_{L_\beta(Q_R)}$. As a consequence

$$|u|_{L^{\text{weak}}_{\alpha(\beta)}(Q_{\sigma R})} \leq C \tag{13}$$

where

$$\alpha_1(\beta) = (\beta-2)\left(1+\frac{p}{N}\right)+m, \qquad \alpha_3(\beta) = \left[\beta\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)+m,$$

$$\alpha_2(\beta) = (\beta-\delta)\left(1+\frac{p}{N}\right)+m, \qquad \alpha_4(\beta) = \beta\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)+m,$$

and

$$\alpha(\beta) = \min\{\alpha_1(\beta), \alpha_2(\beta), \alpha_3(\beta), \alpha_4(\beta)\}. \tag{14}$$

For the iteration, we start by setting

$$\beta_o = \max\{2, m, r\} \tag{15}$$

because the Sobolev embedding theorem and our hypotheses guarantee that $u \in L_{\beta_o,loc}(\Omega_T)$. We shall analyze the sequence of iterations $(\alpha \circ \alpha \circ \cdots \circ \alpha)(\beta_o)$ by cases.

Case 1: α_1. Because we can rewrite $\alpha_1(\beta)$ as

$$\alpha_1(\beta) = \left(1+\frac{p}{N}\right)\beta + (p-2), \tag{16}$$

we see that $\alpha_1(\beta) > \beta$ if and only if

$$\beta > \frac{N}{p}(2-p). \tag{17}$$

As a consequence if $\beta_o > \frac{N}{p}(2-p)$, then the sequence $\beta_o, \alpha_1(\beta_o), \alpha_1(\alpha_1(\beta_o)), \cdots$ will tend to infinity. Indeed, the above shows that the sequence is monotone increasing, so if it tended to a finite limit, that limit would be a fixed point of α_1 greater than β_0. Since there are no such fixed points, we can conclude that the sequence tends to infinity.

That the requirement $\beta_o > \frac{N}{p}(2-p)$ is satisfied is an immediate consequence of H7 and the fact that $\beta \geq r$.

Case 2: α_2. Now $\alpha_2(\beta) > \beta$ if and only if

$$\beta > \delta - \frac{N}{p}(m - \delta). \tag{18}$$

Then because $\beta_o \geq m > \delta > \delta - \frac{N}{p}(m-\delta)$, we conclude that the sequence $\beta_o, \alpha_2(\beta_o), \alpha_2(\alpha_2(\beta_o)), \cdots$ tends to infinity for the same reasons as case 1.

Case 3: α_3. Here the situation is somewhat different. Since $m - (1+\frac{p}{N}) > 0$, we see that the sequence $\beta_o, \alpha_3(\beta_o), \alpha_3(\alpha_3(\beta_o)), \cdots$ tends to infinity provided $(1-\frac{1}{s})(1+\frac{p}{N}) \geq 1$ or equivalently if $s \geq \frac{N+p}{p}$. If not, we see that $\alpha_3(\beta) > \beta$ if and only if

$$\beta < \frac{m - (1+\frac{p}{N})}{1 - (1-\frac{1}{s})(1+\frac{p}{N})} \tag{19}$$

so that $\beta_o, \alpha_3(\beta_o), \alpha_3(\alpha_3(\beta_o)), \cdots$ tends to

$$q_s^* \equiv \frac{m - (1+\frac{p}{N})}{1 - (1-\frac{1}{s})(1+\frac{p}{N})}. \tag{20}$$

Case 4: α_4. This is handled in much the same fashion as case 3. Indeed if $\mu \geq \frac{N+p}{p}$ then $\beta_o, \alpha_4(\beta_o), \alpha_4(\alpha_4(\beta_o)), \cdots$ tends to infinity; otherwise it tends to

$$q_\mu^* \equiv \frac{m}{1 - (1-\frac{1}{\mu})(1+\frac{p}{N})}. \tag{21}$$

Since $q^* = \min\{q_s^*, q_\mu^*\}$, we have our $L_{q,loc}(\Omega_T)$ estimates for $q < \infty$.

3. The $L_{\infty,loc}(\Omega_T)$ Estimates

The boundedness of the solutions shall now be proven using the usual DeGiorgi methods, coupled with an interpolation in the case when $m \leq 2$.

Indeed, let $Q_\rho(x_o, t_o) \Subset \Omega_T$, fix $0 < \sigma < 1$, and let $k > 0$ be chosen later. Then set

$$\rho_n = \sigma\rho + \frac{1-\sigma}{2^n}\rho, \qquad k_n = k\left(1 - \frac{1}{2^{n+1}}\right), \tag{22}$$

and let $Q^n = Q_{\rho_n}(x_o, t_o)$. We now apply (7) where we replace k by k_{n+1}, and Q_R by Q^n, and $Q_{\sigma R}$ by Q^{n+1}. This then gives us

$$\begin{aligned} \left[\iint_{Q^{n+1}} (u - k_{n+1})_+^m \, dx\, dt\right]^{\frac{1}{1+p/N}} &\leq \frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} u - k_{n+1})_+^2 \, dx\, dt \\ &+ \frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} (u - k_{n+1})_+^p \, dx\, dt + \gamma \iint_{Q^n} |u|^\delta \chi[u > k_{n+1}] \, dx\, dt \\ &+ \frac{\gamma 2^n}{(1-\sigma\rho)} \left[\iint_{Q^n} (u - k_{n+1})_+^{\frac{s}{s-1}} dx\, dt\right]^{1-\frac{1}{s}} + \gamma(\operatorname{meas} A_{n+1})^{1-\frac{1}{\mu}} \end{aligned} \tag{23}$$

where

$$A_{n+1} = \{(x,t) \in Q^n : u(x,t) > k_{n+1}\}. \tag{24}$$

Note that

$$\operatorname{meas} A_{n+1} \leq \left(\frac{2^{n+2}}{k}\right)^\theta \iint_{Q^n} (u - k_n)_+^\theta \, dx\, dt \tag{25}$$

for each $\theta \geq 1$; this follows from the fact that

$$\iint_{Q^n} (u - k_n)_+^\theta \, dx \, dt \geq (k_{n+1} - k_n)^\theta \operatorname{meas} A_{n+1}. \tag{26}$$

What happens next depends on the parameter m.

Case 1: $m > 2$. In this instance we shall obtain an iterative inequality for

$$Y_n \equiv ⨌_{Q^n} (u - k_n)_+^m \, dx \, dt = \frac{1}{\operatorname{meas} Q^n} \iint_{Q^n} (u - k_n)_+^m \, dx \, dt \tag{27}$$

with the aid of (23). Indeed,

$$\begin{aligned} &\frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} (u - k_{n+1})_+^2 \, dx \, dt \\ &\qquad \leq \frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \left[\iint_{Q^n} (u - k_{n+1})_+^m \, dx \, dt \right]^{\frac{2}{m}} (\operatorname{meas} A_{n+1})^{1-\frac{2}{m}} \\ &\qquad \leq \frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \left(\frac{2^{n+2}}{k} \right)^{m-2} \iint_{Q^n} (u - k_n)_+^m \, dx \, dt \\ &\qquad \leq \frac{\gamma}{(1-\sigma)^p k^{m-2}} \rho^N 2^{(p+m-2)n} Y_n \end{aligned} \tag{28}$$

while similarly

$$\frac{\gamma 2^{np}}{(1-\sigma)^p \rho^p} \iint_{Q^n} (u - k_{n+1})^p \, dx \, dt \leq \frac{\gamma}{(1-\sigma)^p k^{m-p}} \rho^N 2^{mn} Y_n. \tag{29}$$

Further, because

$$\frac{u(x,t)}{u(x,t) - k_n} \leq \frac{k_{n+1}}{k_{n+1} - k_n} \tag{30}$$

for all $(x,t) \in [u > k_{n+1}]$, we have the estimate

$$\iint_{Q^n} |u|^\delta \chi[u > k_{n+1}] \, dx \, dt \leq \gamma \frac{1}{k^{m-\delta}} \rho^{N+p} 2^{mn} Y_n. \tag{31}$$

Finally, because $\frac{s}{s-1} \leq m$

$$\begin{aligned} &\frac{\gamma 2^n}{(1-\sigma)\rho} \left[\iint_{Q^n} (u - k_{n+1})_+^{\frac{s}{s-1}} \right]^{1-\frac{1}{s}} \\ &\qquad \leq \frac{\gamma}{(1-\sigma) k^{m\left(1-\frac{1}{s}\right)-1}} \rho^{(N+p)\left(1-\frac{1}{s}\right)-1} 2^{\left[m\left(1-\frac{1}{s}\right)-1\right]n} Y_n^{1-\frac{1}{s}}, \end{aligned} \tag{32}$$

and

$$\gamma \left(\operatorname{meas} A_{n+1}\right)^{1-\frac{1}{\mu}} \leq \frac{\gamma}{k^{m\left(1-\frac{1}{\mu}\right)}} \rho^{(N+p)\left(1-\frac{1}{\mu}\right)} 2^{m\left(1-\frac{1}{\mu}\right)n} Y_n^{1-\frac{1}{\mu}}. \tag{33}$$

Combining these results gives us the estimate

$$\begin{aligned}\rho^{N+p}Y_{n+1} \leq & \frac{\gamma\rho^{N+p}2^{(p+m-2)\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{p+N}k^{(m-2)\left(1+\frac{p}{N}\right)}}Y_n^{1+\frac{p}{N}} \\ & + \frac{\gamma\rho^{N+p}2^{(p+m)\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{p+N}k^{(m-p)\left(1+\frac{p}{N}\right)}}Y_n^{1+\frac{p}{N}} \\ & + \frac{\gamma\rho^{(N+p)\left(1+\frac{p}{N}\right)}2^{m\left(1+\frac{p}{N}\right)n}}{k^{(m-\delta)\left(1+\frac{p}{N}\right)}}Y_n^{1+\frac{p}{N}} \\ & + \frac{\gamma\rho^{\left[(N+p)\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)}2^{\left[m\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)n}}{(1-\sigma)^{1+\frac{p}{N}}k^{\left[m\left(1-\frac{1}{s}\right)-1\right]\left(1+\frac{p}{N}\right)}}Y_n^{\left(1-\frac{1}{s}\right)\left(1+\frac{p}{N}\right)} \\ & + \frac{\gamma\rho^{(N+p)\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}2^{m\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)n}}{k^{m\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}}Y_n^{\left(1-\frac{1}{\mu}\right)\left(1+\frac{p}{N}\right)}.\end{aligned} \tag{34}$$

Then, because $m(1-\frac{1}{s})-1>0$, we find that there are constants A and B independent of n and k so that

$$Y_{n+1} \leq AB^nY_n^{1+\frac{p}{N}} + AB^nY_n^{(1-\frac{1}{s})(1+\frac{p}{N})} + AB^nY_n^{(1-\frac{1}{\mu})(1+\frac{p}{N})}. \tag{35}$$

Our assumptions on s and μ imply that $(1-\frac{1}{s})(1+\frac{p}{N})>1$ and $(1-\frac{1}{\mu})(1+\frac{p}{N})>1$, so that standard results on fast geometric convergence imply that $Y_n \to 0$ if Y_o is sufficiently small. By choosing k sufficiently large, we can make Y_o sufficiently small and guarantee that

$$Y_\infty = ⨌_{Q_{\sigma R}} (u-k)_+^m \, dx\, dt = 0, \tag{36}$$

making u bounded above.

Similar considerations for $(u+k)_-$ show that u is bounded below.

Case 2: $m \leq 2$. Let $\lambda > \max\{2, m\}$ be chosen later and set

$$Y_n = ⨌_{Q^n} (u-k_n)_+^\lambda \, dx\, dt; \tag{37}$$

this is well defined thanks to our $L_{q,loc}(\Omega_T)$ estimates. Now for $\Lambda > \lambda > m$, the convexity inequality implies

$$\begin{aligned} Y_{n+1} = & ⨌_{Q^{n+1}} (u-k_{n+1})_+^\lambda \, dx\, dt \\ & \leq \frac{1}{\operatorname{meas} Q^{n+1}} \left(\iint_{Q^{n+1}} (u-k_{n+1})_+^\Lambda \, dx\, dt\right)^{\frac{\lambda}{\Lambda}\theta} \\ & \qquad \times \left(\iint_{Q^{n+1}} (u-k_{n+1})_+^m \, dx\, dt\right)^{\frac{\lambda}{m}(1-\theta)} \end{aligned} \tag{38}$$

where

$$\theta = \frac{\frac{1}{m}-\frac{1}{\lambda}}{\frac{1}{m}-\frac{1}{\Lambda}} = \frac{\Lambda}{\lambda}\frac{\lambda-m}{\Lambda-m}. \tag{39}$$

As a consequence,

$$\iint_{Q^{n+1}} (u-k_{n+1})_+^m \, dx\, dt \geq [\operatorname{meas} Q^{n+1}]^{\frac{\Lambda-m}{\Lambda-\lambda}} \frac{1}{\|u\|_{L_\Lambda(Q_\rho)}^{\frac{\lambda-m}{\Lambda-\lambda}\Lambda}} Y_{n+1}^{\frac{\Lambda-m}{\Lambda-\lambda}} \tag{40}$$

which estimates the left side of (23). The right side is estimated in the same fashion as case 1, so that

$$\frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p} \iint_{Q^n} (u-k_{n+1})_+^2 \, dx\, dt \leq \frac{\gamma \rho^N 2^{(p+\lambda-2)n}}{(1-\sigma)^p k^{\lambda-2}} Y_n, \tag{41}$$

$$\frac{\gamma 2^{np}}{(1-\sigma)^p\rho^p} \iint_{Q^n} (u-k_{n+1})_+^p \, dx\, dt \leq \frac{\gamma \rho^N 2^{\lambda n}}{(1-\sigma)^p k^{\lambda-p}} Y_n, \tag{42}$$

$$\gamma \iint_{Q^n} |u|^\delta \chi[u>k_{n+1}] \, dx\, dt \leq \frac{\gamma \rho^{N+p} 2^{\lambda n}}{k^{\lambda-\delta}} Y_n, \tag{43}$$

further

$$\frac{\gamma 2^n}{(1-\sigma)\rho} \left[\iint_{Q^n} (u-k_{n+1})_+^{\frac{s}{s-1}} \, dx\, dt\right]^{1-\frac{1}{s}} \leq \frac{\gamma \rho^{(N+p)\left(1-\frac{1}{s}\right)} 2^{\left[\lambda\left(1-\frac{1}{s}\right)-1\right]n}}{(1-\sigma) k^{\lambda\left(1-\frac{1}{s}\right)-1}} Y_n^{1-\frac{1}{s}}, \tag{44}$$

and

$$\gamma \left(\operatorname{meas} A_{n+1}\right)^{1-\frac{1}{\mu}} \leq \frac{\gamma \rho^{(N+p)\left(1-\frac{1}{\mu}\right)} 2^{\lambda\left(1-\frac{1}{\mu}\right)n}}{k^{\lambda\left(1-\frac{1}{\mu}\right)}} Y_n^{1-\frac{1}{\mu}}. \tag{45}$$

If we make these substitutions, we shall find constants A and B independent of n and k so that

$$Y_{n+1} \leq AB^n Y_n^{\left(1+\frac{p}{N}\right)\left(\frac{\Lambda-\lambda}{\Lambda-m}\right)} + AB^n Y_n^{\left(1+\frac{p}{N}\right)\left(1-\frac{1}{s}\right)\left(\frac{\Lambda-\lambda}{\Lambda-m}\right)} + AB^n Y_n^{\left(1+\frac{p}{N}\right)\left(1-\frac{1}{\mu}\right)\left(\frac{\Lambda-\lambda}{\Lambda-m}\right)}. \tag{46}$$

Then because

$$\lim_{\Lambda\to\infty} \frac{\Lambda-\lambda}{\Lambda-m} = 1 \tag{47}$$

we can choose Λ and λ sufficiently large that both $(1+\frac{p}{N})(1-\frac{1}{s})(\frac{\Lambda-\lambda}{\Lambda-m}) > 1$ and $(1+\frac{p}{N})(1-\frac{1}{\mu})(\frac{\Lambda-\lambda}{\Lambda-m}) > 1$. The usual results on fast geometric convergence let us proceed as we did in case 1, giving us our result.

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